Testing Matrix Product States

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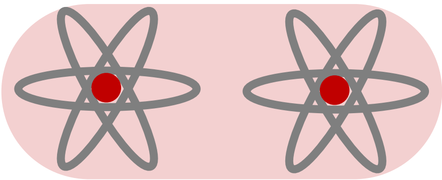
Joint work with
John Wright (UC Berkeley)

arxiv: 2201.01824

Our work is about

Testing Entanglement in pure quantum states



State of 2 qudits

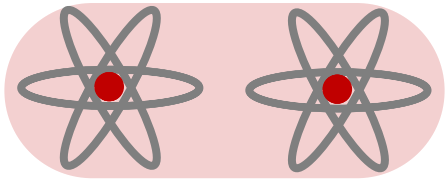
is a vector in tensor product space $\mathbb{C}^d \otimes \mathbb{C}^d$

$$|\psi\rangle = \sum_{i_1, i_2 = 1}^{d} a_{i_1 i_2} \cdot |i_1\rangle\, |i_2\rangle$$

$$a_{i_1 i_2} \in \mathbb{C}, \qquad \| |\psi\rangle \|_2 = 1$$

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

Product state



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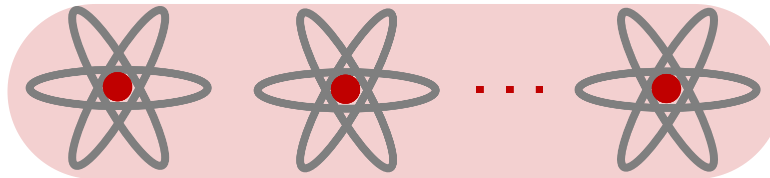
$$a_{i_1 i_2} \in \mathbb{C}, \qquad \||\psi\rangle\|_2 = 1$$

$$|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$$

Entangled state



State of $\boldsymbol{n}$ **qudits**

is a vector in tensor product space $\left(\mathbb{C}^{d}\right)^{\otimes \boldsymbol{n}}$

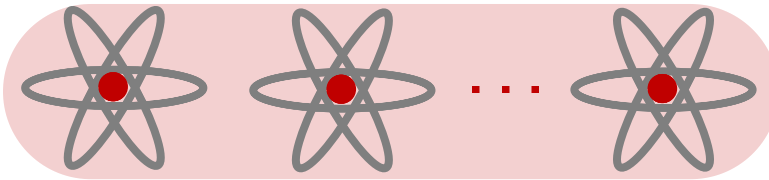
$$|\boldsymbol{\psi}\rangle = \sum_{i_1,\ldots,i_n=1}^{d} a_{i_1\ldots i_n} |i_1\rangle \cdots |i_n\rangle$$

$$a_{i_1\ldots i_n} \in \mathbb{C} \quad \| |\boldsymbol{\psi}\rangle \|_2 = 1$$

Again state $|\boldsymbol{\psi}\rangle$ can be product or entangled:

$$|\boldsymbol{\psi}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \cdots \otimes |\psi_n\rangle$$

Product state



State of n **qudits**

is a vector in tensor product space $\left(\mathbb{C}^{d}\right)^{\otimes n}$

$$|\psi\rangle = \sum_{i_1,\dots,i_n=1}^{d} a_{i_1\dots i_n} |i_1\rangle \cdots |i_n\rangle$$

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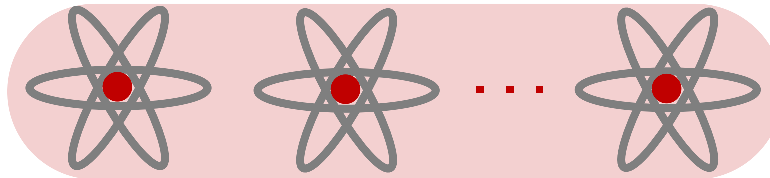
Entangled state



Questions we can ask:

Is a state $|\psi\rangle$ entangled or product?

How entangled is a state $|\psi\rangle$?

Long history in quantum information:

Bell test or quantum games

Tensor optimization

Quantum cryptography

Hamiltonian complexity

Quantum many-body physics

This talk:

Statistical theory of many-body entanglement

Property testing model

Entanglement tester is an algorithm $\mathcal{A}$ such that

1. If $|\psi\rangle$ has at most certain amount of entanglement

$$\Pr\left[\mathcal{A} \text{ accepts given } |\psi\rangle^{\otimes m}\right] \geq 2/3$$ *Completeness*

2. If $|\psi\rangle$ is far from states with at most certain amount of entanglement

$$\Pr\left[\mathcal{A} \text{ accepts given } |\psi\rangle^{\otimes m}\right] \leq 1/3$$ *Soundness*

What is the fewest number of copies m needed for entanglement testing?

One way to quantify many-body entanglement

$\mathrm{MPS}(r)$: Matrix product states with bond dimension r

$$|\psi\rangle = \sum_{i_1,i_2=1}^{d} \mathrm{tr}\left[A_i\, B_j\right] \cdot |i\rangle\, |j\rangle$$

$A_1, A_2, \ldots, A_d$
$B_1, B_2, \ldots, B_d$
$r \times r$ complex matrices

Example:

$$|\psi\rangle = \left(\sum_{i=1}^{d} a_i \cdot |i\rangle\right) \otimes \left(\sum_{j=1}^{d} b_j \cdot |j\rangle\right) \qquad a_i, b_j \in \mathbb{C}$$

$|\psi\rangle$ is a product state

$$A_i = a_i, B_j = b_j$$

Bond dimension $r = 1$

$\mathrm{MPS}(r)$: Matrix product states with bond dimension r

$$|\psi\rangle = \sum_{i_1,i_2=1}^{d} \mathrm{tr}\left[A_i\, B_j\right] \cdot |i\rangle\, |j\rangle$$

$A_1, A_2, \dots, A_d$
$B_1, B_2, \dots, B_d$
$r \times r$ complex matrices

Example: $$|\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle|1\rangle + \frac{1}{\sqrt{2}}|2\rangle|2\rangle$$

$$A_1 = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, A_3 = 0, \dots, A_d = 0$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}, B_3 = 0, \dots, B_d = 0$$

Bond dimension $r = 2$

$\mathrm{MPS}(r)$: Matrix product states with bond dimension r

$$|\psi\rangle = \sum_{i_1,i_2=1}^{d} \mathrm{tr}\left[A_i\, B_j\right] \cdot |i\rangle\, |j\rangle$$

$A_1, A_2, \dots, A_d$
$B_1, B_2, \dots, B_d$
$r{\times}r$ complex matrices

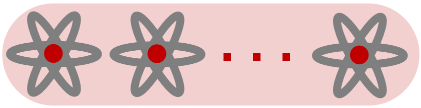
Example: $$|\psi\rangle = \frac{1}{\sqrt{d}}\,|1\rangle|1\rangle + \cdots + \frac{1}{\sqrt{d}}\,|d\rangle|d\rangle$$

Needs bond dimension $r = d$

If $r = d$, any state $|\psi\rangle$ can be written as an MPS

Bond dim limits the amount of entanglement

MPS(r): **Matrix product states**
with bond dimension r



$$|\psi\rangle = \sum_{i_1,\ldots,i_n=1}^{d} \mathrm{tr}\left[A_{i_1}^{(1)} \cdots A_{i_n}^{(n)}\right] \cdot |i_1\rangle \cdots |i_n\rangle$$

$$\begin{matrix} A_1^{(1)}, A_2^{(1)}, \ldots, A_d^{(1)} \\ A_1^{(2)}, A_2^{(2)}, \ldots, A_d^{(2)} \\ \vdots \\ A_1^{(n)}, A_2^{(n)}, \ldots, A_d^{(n)} \end{matrix}$$

$r \times r$ **complex matrices**

If $r \sim d^n$, any state $|\psi\rangle$ can be written as an MPS

Bond dim limits the amount of entanglement

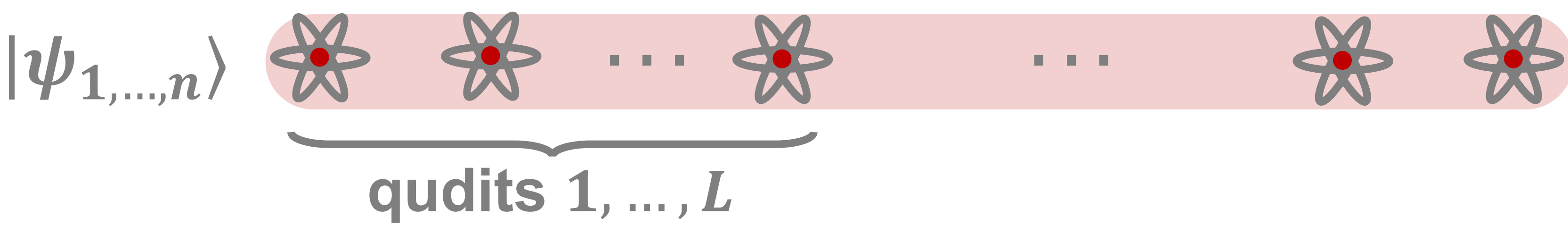
Many states ψ of interest in physics are MPS of small bond dimension r

Alternative characterization of MPS

$\mathrm{MPS}(r)$: **Matrix product states**
with bond dimension r

Another view of $\mathrm{MPS}(r)$ in terms of:

Reduced state $\rho_{1,\dots,L} = \mathrm{tr}_{L+1,\dots,n}\, |\psi_{1,\dots,n}\rangle\langle\psi_{1,\dots,n}|$



$\mathrm{MPS}(r)$: **Matrix product states**
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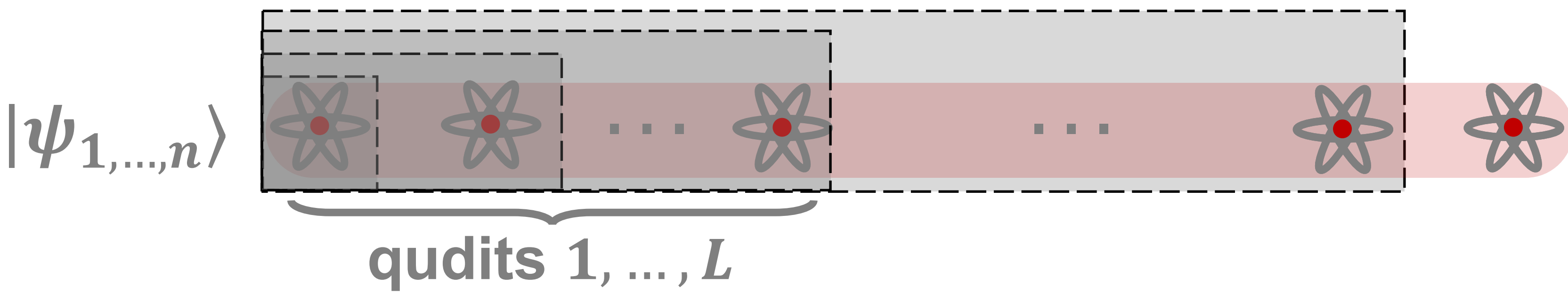

$\mathrm{rank}(\rho_{1,\dots,L}) \leq r$ **for** $1 \leq L \leq n$

Let’s go back to testing entanglement

Property testing model

MPS tester is an algorithm $\mathcal{A}$ such that

1. If $|\psi\rangle \in \mathrm{MPS}(r)$ then
$$\Pr\left[\mathcal{A} \text{ accepts given } |\psi\rangle^{\otimes m}\right] \geq 2/3$$ *Completeness*
2. If $\mathrm{Dist}_r(|\psi\rangle) \geq \delta$ then
$$\Pr\left[\mathcal{A} \text{ accepts given } |\psi\rangle^{\otimes m}\right] \leq 1/3$$ *Soundness*

What does it mean for $|\psi\rangle$ to be far from $\mathrm{MPS}(r)$?

$$\mathrm{Overlap}_r(|\psi\rangle) = \max_{|\phi\rangle \in \mathrm{MPS}(r)} |\langle\psi|\phi\rangle|^2$$

$$\mathrm{Dist}_r(|\psi\rangle) = \min_{|\phi\rangle \in \mathrm{MPS}(r)} \mathrm{D}_{\mathrm{trace}}(\psi, \phi) = \min_{|\phi\rangle \in \mathrm{MPS}(r)} \sqrt{1 - |\langle\psi|\phi\rangle|^2}$$

$$\mathrm{Dist}_r(|\psi\rangle) = \sqrt{1 - \mathrm{Overlap}_r(|\psi\rangle)}$$

Property testing model

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Goal: Finding the smallest number of copies m

for a given
- number of qudits n
- bond dimension r
- distance δ

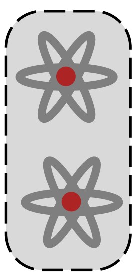
MPS testing when $r = 1$
(Product testing)

Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

[Mintert, Kuś, Buchleitner]
[Harrow and Montanaro]

Accept ↓ Reject ↓

- Measure $\{\Pi_{\mathrm{SWAP}}, \mathrm{I} - \Pi_{\mathrm{SWAP}}\}$ *on all pairs of qudits in* $|\psi\rangle^{\otimes 2}$



SWAP test

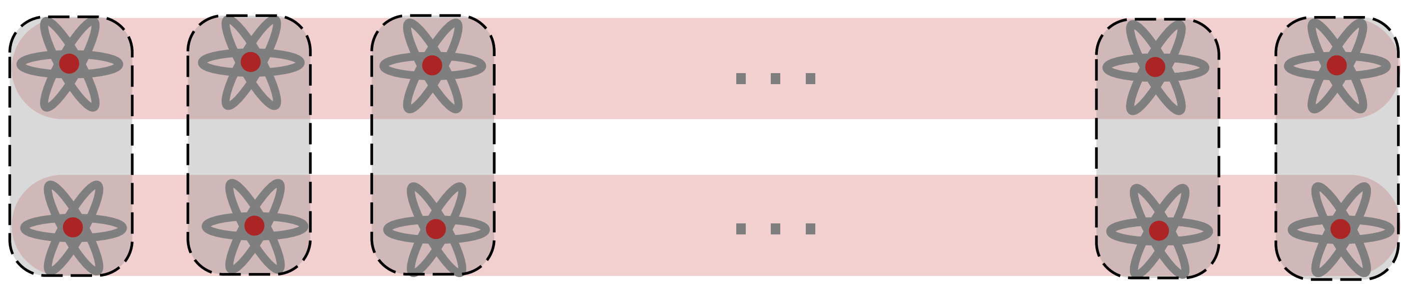
$$\Pi_{\mathrm{SWAP}} = \frac{\mathrm{I+SWAP}}{2}, \quad \mathrm{I} - \Pi_{\mathrm{SWAP}} = \frac{\mathrm{I-SWAP}}{2}$$

$$\Pr[\text{SWAP test accepts } \rho \otimes \rho] = \frac{1}{2} + \frac{1}{2}\mathrm{tr}[\rho^2]$$ ← Purity

- ***Accept if all SWAP tests accept***

$\psi_{1,\dots,n}$

$\psi_{1,\dots,n}$



Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

[HM13]:

Product states pass this test with probability 1

$$|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_k\rangle \otimes \cdots \otimes |\psi_n\rangle$$
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$$\Pr\left[\text{SWAP test accepts } |\psi_k\rangle^{\otimes 2}\right] = \frac{1}{2} + \frac{1}{2}|\langle\psi_k|\psi_k\rangle|^2 = 1$$

Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

[HM13]:

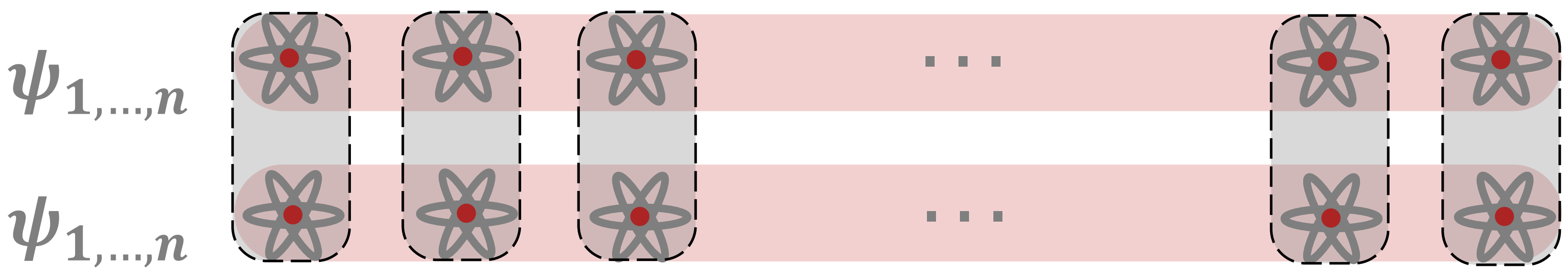
Product states pass this test
with probability 1

States δ-far from product fail this test
with probability $\Omega(\delta^2)$

Why?

Entangled $|\psi_{1,\ldots,n}\rangle$ means some mixed subsystems with $\mathrm{tr}[\rho^2] < 1$

$$\Pr[\text{SWAP test accepts } \rho \otimes \rho] = \frac{1}{2} + \frac{1}{2}\mathrm{tr}[\rho^2] < 1$$



Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

[HM13]:

Product states pass this test
with probability $\mathbf{1}$

States $\boldsymbol{\delta}$***-far from product fail this test***
with probability $\Omega(\delta^2)$

Rejection probability can be boosted to $2/3$
by repeating on $m = O\left(\frac{1}{\delta^2}\right)$ **pairs**

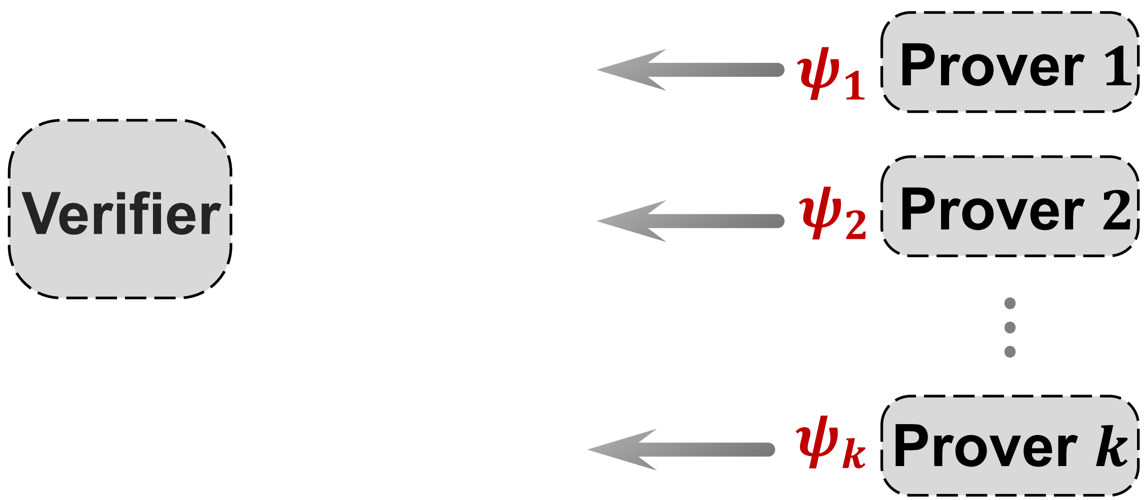
Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

[HM13]:

Product states pass this test
with probability 1

States δ-*far from product fail this test*
with probability $\Omega(\delta^2)$

This implies $\mathrm{QMA}(k) = \mathrm{QMA}(2)$ for $k \geq 2$ [HM13]



Product test (testing $\mathrm{MPS}(r)$ with $r = 1$)

[HM13]:

Product states pass this test with probability 1

States δ***-far from product fail this test with probability*** $\Omega(\delta^2)$

This implies $\mathrm{QMA}(k) = \mathrm{QMA}(2)$ **for** $k \geq 2$ **[HM13]**

With applications in hardness of tensor optimization problems

Open problem of [HM13] and [MdW13]:

Can proof of product test be simplified and improved?

Result 1

Improved and simple analysis of product test

Proof sketch of product test

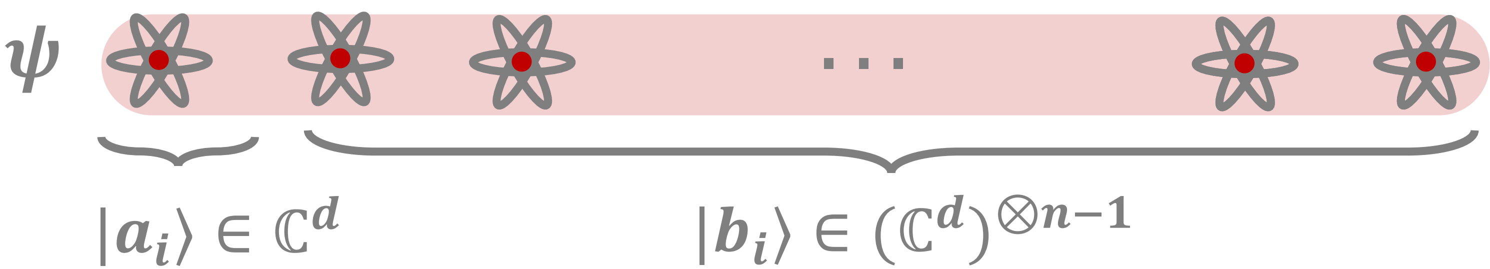
Schmidt decomposition

$$|\psi\rangle = \sqrt{\lambda_1}\,|a_1\rangle|b_1\rangle + \sqrt{\lambda_2}\,|a_2\rangle|b_2\rangle + \cdots + \sqrt{\lambda_d}\,|a_d\rangle|b_d\rangle$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$

$\lambda_1 + \lambda_2 + \cdots \lambda_d = 1$

$|a_i\rangle \in \mathbb{C}^d, |b_i\rangle \in (\mathbb{C}^d)^{\otimes n-1}$



Proof sketch of product test

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$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$

$\lambda_1 + \lambda_2 + \cdots \lambda_d = 1$

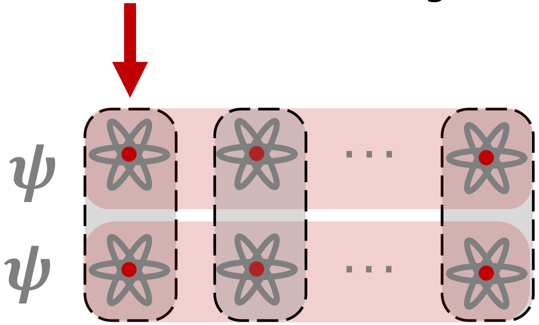
$|a_i\rangle \in \mathbb{C}^d, |b_i\rangle \in (\mathbb{C}^d)^{\otimes n-1}$

Suppose $|\psi\rangle$ is far from product.

If λ_1 *is small*:

1st qudit is highly entangled with the rest

1st SWAP test rejects with good probability



Proof sketch of product test

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$|a_i\rangle \in \mathbb{C}^d, |b_i\rangle \in (\mathbb{C}^d)^{\otimes n-1}$

Suppose $|\psi\rangle$ is far from product.

If λ_1 *is large:*

$|\psi\rangle \approx |a_1\rangle \otimes |b_1\rangle$ ***and 1st SWAP test accepts***

But for **$|\psi\rangle$ to be far from product**

$|b_1\rangle$ ***has to be far from product***

Remaining SWAP tests reject with high prob. (by induction)

Proof sketch of product test

Schmidt decomposition

$$|\psi\rangle = \sqrt{\lambda_1}\,|a_1\rangle|b_1\rangle + \sqrt{\lambda_2}\,|a_2\rangle|b_2\rangle + \cdots + \sqrt{\lambda_d}\,|a_d\rangle|b_d\rangle$$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$

$\lambda_1 + \lambda_2 + \cdots \lambda_d = 1$

$|a_i\rangle \in \mathbb{C}^d, |b_i\rangle \in (\mathbb{C}^d)^{\otimes n-1}$

Given $|\psi\rangle$ that is δ-far from product states,

$$\Pr[\text{Product test rejects } |\psi\rangle^{\otimes 2}] \geq \begin{cases} \delta^2 - \delta^4 & \delta \leq \sqrt{1/2} \\ \frac{2}{3}\delta^2 - \frac{1}{3}\delta^4 & \text{otherwise} \end{cases}$$

Our bound in tight for $n \geq 2, \delta \leq \sqrt{1/2}$ as shown by

$$|\psi\rangle = \sqrt{1-\delta^2}\,|1\rangle|1\rangle + \delta\,|2\rangle|2\rangle$$

Result 2

Testing $\mathrm{MPS}(r)$ with $r \geq 2$
Upper bound and Lower bound

Testing $\mathrm{MPS}(r)$ with $r \geq 2$

Main ingredient is the rank tester of O'Donnell and Wright [OW15]

Tests if $\mathrm{rank}(\rho) \leq r$ or ρ is ϵ-far from rank-r states using $m = \Theta(r^2/\epsilon)$ copies

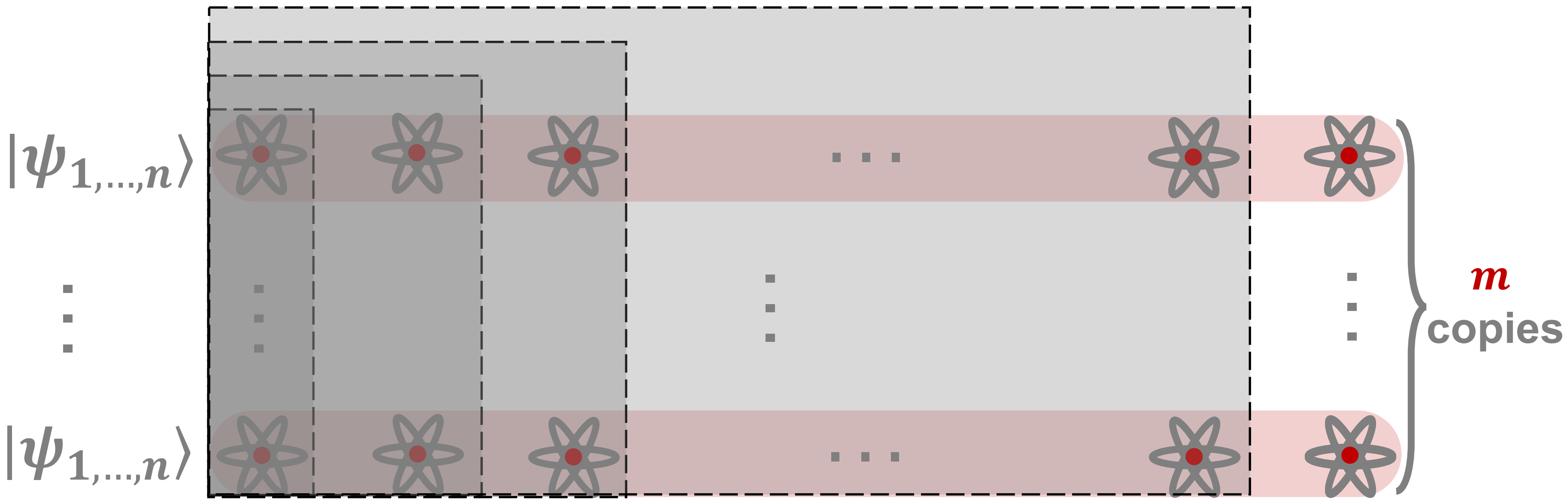
Uses generalized SWAP test called weak Schur sampling

Optimal test with perfect completeness and $O(1)$ soundness

Can be performed with a quantum circuit of size $\mathrm{poly}(m, \log d)$

[Krovi19], [Harrow05]

Testing $\mathrm{MPS}(r)$ with $r \geq 2$



Our MPS tester

1) Simultaneously performs rank tester $\{\Pi_{\leq r}, I - \Pi_{\leq r}\}$ on qudits $1, \dots, L$ for each $1 \leq L \leq n$

2) Rejects if any of the rank testers reject

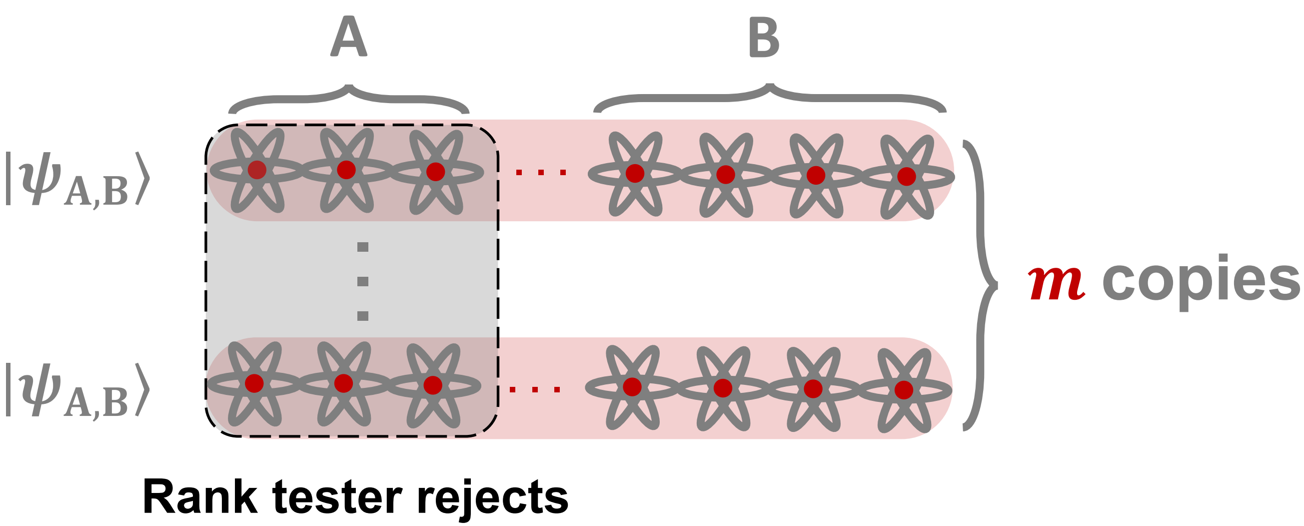
Testing $\mathrm{MPS}(r)$ with $r \geq 2$

Upper bound: Our MPS tester requires $m = O(nr^2/\delta^2)$

Proof relies on

1) $\exists$Cut (A, B) where ρ_A is $\Omega(\delta^2/n)$-far from being rank r

$\Longrightarrow$ ***Rank tester with*** $m = O(nr^2/\delta^2)$ **detects this w.h.p**



Testing $\mathrm{MPS}(r)$ with $r \geq 2$

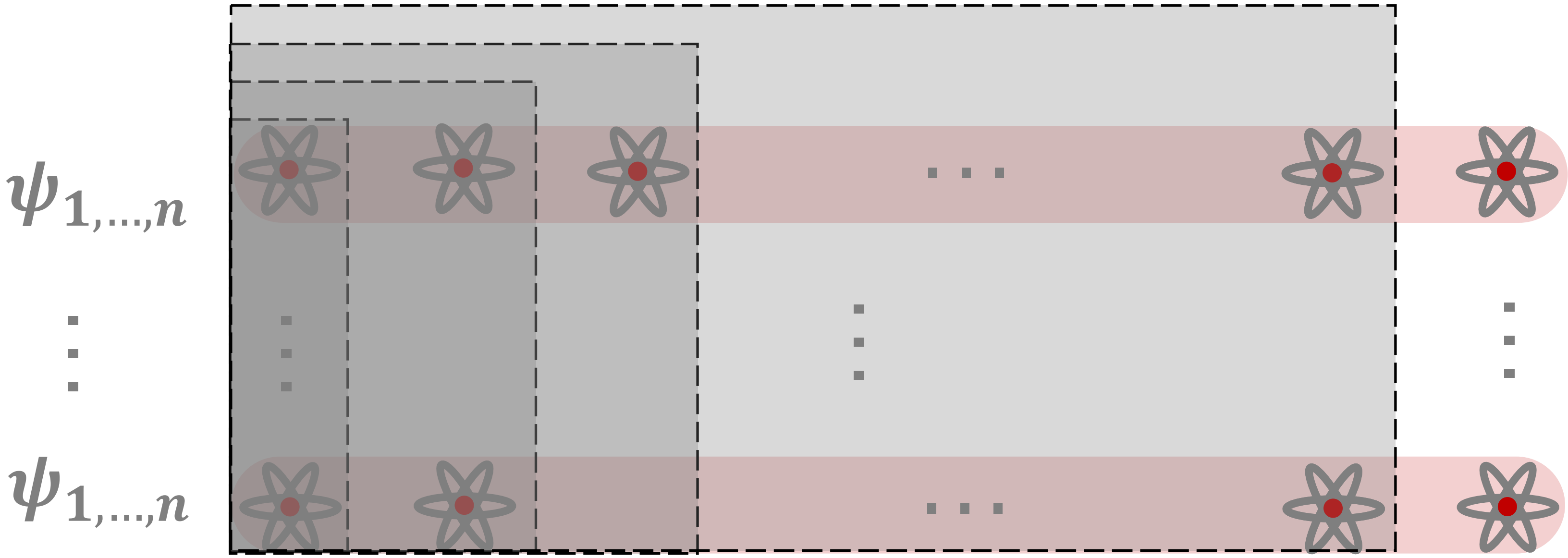
Upper bound: Our MPS tester requires $m = O\left(nr^2/\delta^2\right)$

Proof relies on

1) $\exists$Cut (A, B) where ρ_A is $\Omega\left(\delta^2/n\right)$-far from being rank r

$\Longrightarrow$ ***Rank tester with*** $m = O\left(nr^2/\delta^2\right)$ detects this w.h.p

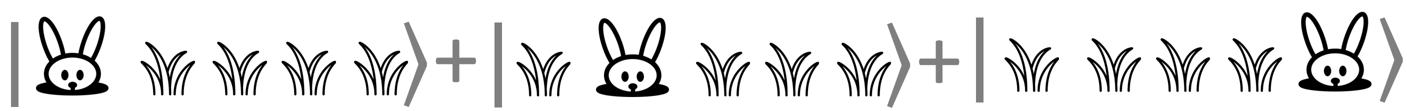
2) The rank tester projectors mutually commute



Testing $\mathrm{MPS}(r)$ with $r \geq 2$

Can this analysis of be improved to show $m = O(1)$ copies are sufficient?

Can be done for the "bunny state":

$|b_n\rangle = \frac{1}{\sqrt{n-1}}$ 

- $|b_n\rangle \in \mathrm{MPS}(3)$ and $\sqrt{1/3}$-far from $\mathrm{MPS}(2)$
- The $r = 2$ MPS tester rejects $|b_n\rangle^{\otimes 3}$ with probability $\geq \frac{1}{6}$

Testing $\mathrm{MPS}(r)$ with $r \geq 2$

Can this analysis of be improved
to show $m = O(1)$ copies are sufficient?

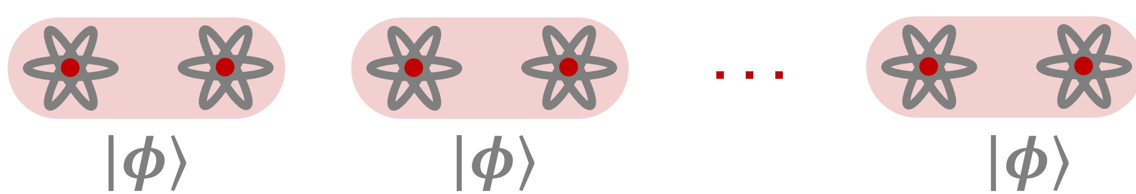
Can't be done for general states!

Lower bound:

Any MPS tester requires $m = \Omega\left(\sqrt{n}/\delta^2\right)$

The hard example: $|\psi\rangle$ **and its random local rotations**

where $|\phi\rangle$ **is** $1/\sqrt{n}$**-far from** $\mathrm{MPS}(r)$

$|\psi\rangle = |\phi\rangle^{\otimes n/2} =$ 

Testing $\mathrm{MPS}(r)$ with $r \geq 2$

Lower bound: Any MPS tester requires $m = \Omega(\sqrt{n}/\delta^2)$

Proof relies on

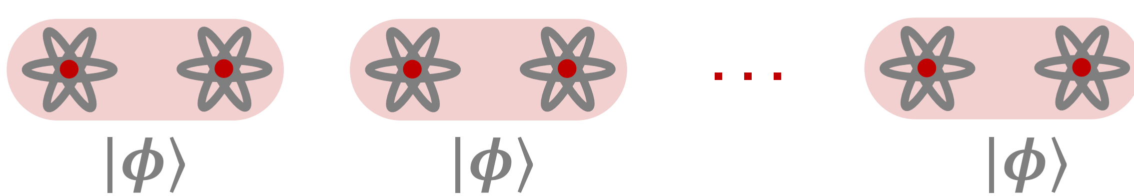
1) $\mathrm{Overlap}_r(|\phi\rangle) = \omega$ then $\mathrm{Overlap}_r(|\phi\rangle^{\otimes n/2}) = \omega^{n/2}$

$\omega \sim 1 - 1/n$ then $\mathrm{Overlap}_r(|\phi\rangle^{\otimes n/2}) \sim 1/2$

2) $\exists |\gamma\rangle \in \mathrm{MPS}(r)$ such that unless $m = \Omega(\sqrt{n})$

$$\left(\mathbb{E}_{U,V}\left(U \otimes V \cdot |\phi\rangle\langle\phi| \cdot U^\dagger \otimes V^\dagger\right)^{\otimes m}\right)^{\otimes n/2}$$

$$\approx \left(\mathbb{E}_{U,V}\left(U \otimes V \cdot |\gamma\rangle\langle\gamma| \cdot U^\dagger \otimes V^\dagger\right)^{\otimes m}\right)^{\otimes n/2}$$

$|\psi\rangle = |\phi\rangle^{\otimes n/2} =$ 

Summary

Developed algorithms for testing matrix product states

1) Simple and improved analysis of the product test

2) Upper bound of $O(n)$ for MPS testing with bond dim ≥ 2

3) Lower bound of $\Omega(\sqrt{n})$ for MPS testing with bond dim ≥ 2

Open questions

1) Optimal copy complexity of $\mathrm{MSP}(r)$ testing for $r \geq 2$

2) Testing more general entangled states e.g. tensor networks

3) Testing if a mixed quantum state is separable (unentangled)

Testing Matrix Product States

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Joint work with
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arxiv: 2201.01824